Heat transport in a liquid layer locally heated on its free surface

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A strong heat flux, localized on the upper surface of a fluid, sets up strong convection motions through thermocapillary forces, which limits the temperature elevation in the pool, therefore limiting the efficiency in fusion welding processes. We propose a theoretical estimate of the temperature elevation when the fluid motion is laminar or turbulent, the weld pool surface remaining flat. Our treatment follows the theoretical work of Shraiman and Siggia [Phys. Rev. A **42**, 3650 (1990)] in Rayleigh-Bénard convection. In the laminar case, the temperature elevation is proportional to the incident power to the $\frac{3}{4}$ power, in agreement with earlier estimates, and in the turbulent case, to the incident power to the $\frac{2}{3}$ power. [S1063-651X(96)51311-5]

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Heat transport by convection is a ubiquitous phenomenon in nature [1] or in industrial processes [2,3]. The paradigm of Rayleigh-Benard convection, whereby fluid is heated from below in a closed container, has been studied in great detail, both in regimes close to the convection threshold [4], and far from threshold. In the latter case, fluid motion becomes highly disorganized (turbulent) [5]. Well controlled experiments have shown that the Nusselt number, Nu, measuring the dimensionless heat flux behaves as a function of the Rayleigh number, Ra (the dimensionless temperature across the cell)

$$\mathrm{Nu} \propto \mathrm{Ra}^{2/7} \tag{1}$$

over more than five decades of Rayleigh number [6,7]. This result is at odds with earlier predictions [8]. The flow exhibits thin turbulent boundary layers along the walls, with a superimposed large scale coherent motion over the entire cell. Theoretical work, properly taking into account these features of the flow, provides an explanation of the Nusselt-Rayleigh relation, Eq. (1), as well as the other scaling relations found experimentally [5,9].

In welding or metal evaporation processes [2,3], an intense localized source of heat is applied at the free surface of a metal, thereby melting the metal and setting up convective motions in the liquid. Surface forces (Marangoni effect) are known to be more important in this configuration than bulk forces (buoyancy) [2,10]. The relation between the temperature elevation in the melt and the injected power in the system (the equivalent of the Nusselt-Rayleigh relation) is important in a number of practical applications. The purpose of this Rapid Communication is to provide theoretical estimates of this relation. Our approach follows the theoretical work reviewed in Ref. [5]. The laminar problem is considered first, and the results agree well with a numerical solution of the equations of motion, and with the theoretical predictions of Chan, Chen and Mazumder [11]. In the turbulent regime, a few reasonable assumptions on the flow are then necessary to obtain a prediction for the temperature elevation as a function of the heating power.

We assume throughout this work that the motion in the fluid is described by the Boussinesq equations [12]:

$$\rho[\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}] = -\boldsymbol{\nabla} p + \rho\alpha\mathbf{g}T + \rho\nu\boldsymbol{\nabla}^2\mathbf{u}, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \tag{3}$$

$$\partial_t T + (\mathbf{u}\cdot\boldsymbol{\nabla})T = \kappa\boldsymbol{\nabla}^2 T, \tag{4}$$

where $\mathbf{u}$ and T are the velocity and the temperature fields, ρ the mean density, ν the viscosity, κ the thermal diffusivity, and α the coefficient of expansion of the fluid, which are all assumed to be constant. The upper surface is assumed to be flat, even right under the incident heating beam. This assumption is in our view the most serious limitation of the approach presented here. The melt is assumed to fill a rectangular [two-dimensional (2D)] or cylindrical (3D) box, of depth H and of radius R (in 3D). The incident heating flux is confined to a limited region, resulting in the boundary condition on the upper interface ($z=0$),

$$\kappa\partial_z T = Q_0 q(r), \tag{5}$$

where $q(r)$ is a function of order 1, which vanishes for $r \geqslant r_0$, the beam radius, and Q_0 is the injected power in the system, divided by the heat capacity in the system, assumed constant. We take r_0, R, and H to be of the same order of magnitude. Because of the horizontal temperature gradient on the fluid interface, the Marangoni effect induces a shear stress described by

$$\rho\nu\partial_z \mathbf{u}_\parallel - \left(\frac{d\sigma}{dT}\right)\boldsymbol{\nabla}_\parallel T = \mathbf{0}, \tag{6}$$

where $(d\sigma/dT)$ is the derivative of the surface tension with respect to temperature. In the applications we have in mind, $(d\sigma/dT)$ is negative, so the Marangoni effect tends to generate a flow from the hot to the cold regions. As a consequence, in the geometry considered here, flow is pushed away from the center along the free surface.

It is convenient to use dimensionless variables defined by $\overline{x} \equiv x/H$ (space), $\overline{t} \equiv \kappa t/H^2$ (time), $\overline{\mathbf{u}} \equiv \mathbf{u}H/\kappa$ (velocity), $\overline{p} \equiv pH^2/\rho\kappa^2$ (pressure), and $\theta \equiv (T-T_M)\kappa/(HQ_0)$ (temperature). The equations of motion read

$$\frac{1}{\mathrm{Pr}}(\partial_{\bar{t}}\overline{\mathbf{u}}+(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}+\nabla\overline{p})=\nabla^2\overline{\mathbf{u}}+\mathrm{Ra}\,\theta\hat{\mathbf{z}}, \tag{7}$$

$$\nabla\cdot\overline{\mathbf{u}}=0, \tag{8}$$

$$\partial_{\bar{t}}\overline{\theta}+(\overline{\mathbf{u}}\cdot\nabla)\overline{\theta}=\nabla^2\overline{\theta}, \tag{9}$$

where $\mathrm{Pr}\equiv\nu/\kappa$ is the Prandtl number and $\mathrm{Ra}\equiv g\alpha Q_0H^4/\nu\kappa^2$ is the Rayleigh number. In addition, the boundary condition Eq. (6) becomes

$$\partial_{\bar{z}}\overline{\mathbf{u}}_{\parallel}+\mathrm{Ma}\nabla_{\parallel}\overline{\theta}=\mathbf{0}, \tag{10}$$

where $\mathrm{Ma}\equiv-d\sigma/dT(Q_0H^2/\kappa^2\mu)$ is the Marangoni number (here, $\mathrm{Ma}>0$). The temperature in dimensional units, θ, is equal to the dimensionless temperature $\overline{\theta}$, multiplied by Q_0 (the heat power injected in the system). In the following, we work with dimensionless variables only and for simplicity, we drop the overbars.

In a statistically steady state, the following equalities result from the equations of motion and the boundary conditions in the problem:

$$\int_{\mathrm{vol}}(\nabla\theta)^2d^3x=\int_{\mathrm{surface}}\theta q\,d^2x \tag{11}$$

and

$$\sum_{i,j}\int_{\mathrm{vol}}(\partial_iu_j)^2d^3x+\mathrm{Ma}\int_{\mathrm{surface}}(\mathbf{u}_{\parallel}\cdot\nabla_{\parallel})\theta d^2x$$
$$=-\mathrm{Ra}\left(\int_{\mathrm{surface}}z\nabla\theta\cdot dS-\int\theta dS\right). \tag{12}$$

Equation (11) expresses the balance between thermal dissipation $(\nabla\theta)^2$ and production, whereas Eq. (12) relates the dissipation of kinetic energy $(\partial_iu_j)^2$ to the source of motion, through the Marangoni effect (Ma term) and buoyancy (Ra term). As pointed out already, the effect of buoyancy happens to be negligible compared to the Maragoni effect. This was explicitly checked numerically, for a realistic set of numerical values. We will restrict ourselves to the purely Marangoni case ($\mathrm{Ra}=0$).

We begin by considering the laminar regime. Our numerical results, obtained with TRIO [13], show the existence of a large recirculation zone, extending over the half cell, the streamlines being concentrated right underneath the interface. The temperature is almost constant everywhere, except in a narrow region under the incident energy flux.

To estimate the temperature elevation as a function of the heating power, we will assume that the dissipation and production terms are localized in a narrow boundary layer of size Δ_θ (thermal boundary layer), and Δ_U (viscous boundary layer) near the upper surface. The magnitudes of the maximum velocity and temperature in these boundary layers are denoted by U and Θ, respectively. The thermal dissipation term comes essentially from the boundary layers, and is estimated to be

$$\int_{\mathrm{vol}}(\nabla\theta)^2d^3x\approx\frac{\Theta^2}{\Delta_\theta^2}\times\Delta_\theta\times(\mathrm{surface})\approx\frac{\Theta^2}{\Delta_\theta}, \tag{13}$$

so Eq. (11) yields

$$\Theta\sim\Delta_\theta. \tag{14}$$

Similarly, the dissipation of mechanical energy can be estimated by

$$\sum_{i,j}\int_{\mathrm{vol}}(\partial_iu_j)^2d^3x\sim\frac{U^2}{\Delta_U}. \tag{15}$$

As a result, Eq. (12) leads to

$$U\sim\mathrm{Ma}\Theta\Delta_U. \tag{16}$$

Equations (13) and (16) provide two relations between the quantities characterizing the flow in the boundary layers. No assumption has been made yet about their relative sizes. To proceed one needs to treat separately the low and high Prandtl number cases. The low Prandtl number, appropriate in the case of liquid metals ($\mathrm{Pr}\sim10^{-2}$) leads to a viscous boundary layer much thinner than the thermal one: $\Delta_U\ll\Delta_\theta$. As a consequence, the temperature does not vary across the viscous boundary layer. The width of the viscous boundary layer can be simply estimated by equilibrating the viscous and the nonlinear term in Eq. (7), which leads to

$$U\Delta_U^2\sim\mathrm{Pr}. \tag{17}$$

Similarly, the width of the thermal boundary layer is obtained by balancing the diffusive term with the advection term in Eq. (10). The diffusion term is estimated to be of order $\sim\Theta/\Delta_\theta^2$. The thermal boundary layer is sensitive to the upward jets of fluids underneath the incident beam, but not to the (small scale) details of the flow in the viscous boundary layers. The order of magnitude of the flow in the vertical jet in the center of the cell, U_v, is estimated by incompressibility considerations. The flux of fluid underneath the beam is of order $U_v\times$(surface of the beam). This flux must be equal to the flux generated by the Marangoni effect at the free surface, which is estimated to be $U\times\Delta_U$. Comparing these two fluxes leads to $U_v\sim U\Delta_U$. The order of magnitude of the $(\mathbf{u}\cdot\nabla)\theta$ term in Eq. (10) is therefore $\sim U_v\Theta/\Delta_\theta$, and balancing with the diffusive term leads to

$$\Delta_\theta\sim U^{-1}\Delta_V^{-1}. \tag{18}$$

Equations (13,16–18) yield in turn

$$\Theta\sim\mathrm{Pr}^{-1/2}\mathrm{Ma}^{-1/4}, \tag{19a}$$

$$U\sim\mathrm{Ma}^{1/2}, \tag{19b}$$

$$\Delta_U\sim\mathrm{Pr}^{1/2}\mathrm{Ma}^{-1/4}, \tag{19c}$$

$$\Delta_\theta\sim\mathrm{Pr}^{-1/2}\mathrm{Ma}^{-1/4}. \tag{19d}$$

As a consequence of Eq. (19a), the temperature elevation under the beam scales as the heat power injected, to the power $\frac{3}{4}$. This in turn implies that the Nusselt number, the ratio between the heat flux injected and the heat flux necessary to maintain the system at the same temperature in a diffusive regime, behaves like

$$\mathrm{Nu}\sim\mathrm{Ma}^{1/4}\mathrm{Pr}^{1/2}. \tag{20}$$

In the large Prandtl number case the viscous boundary layer is much thicker than the thermal boundary layer: $\Delta_U \gg \Delta_\theta$. Equation (17), expressing the balance between inertial and dissipative terms in the viscous boundary layers, still holds in this case. The thin thermal boundary layer is sensitive to a stagnation point flow very close to the upper surface: $u_r \sim Ur$, $u_z \sim -Uz$. As a result, the balance between advection and diffusion in the thermal boundary layer leads to

$$U\Delta_\theta^2 \sim 1. \tag{21}$$

From Eqs. (13,16–18,21), one obtains

$$\Theta \sim \mathrm{Pr}^{-1/8}\mathrm{Ma}^{-1/4}, \tag{22a}$$

$$U \sim \mathrm{Pr}^{1/4}\mathrm{Ma}^{1/2}, \tag{22b}$$

$$\Delta_U \sim \mathrm{Pr}^{3/8}\mathrm{Ma}^{-1/4}, \tag{22c}$$

$$\Delta_\theta \sim \mathrm{Pr}^{-1/8}\mathrm{Ma}^{-1/4}. \tag{22d}$$

As was the case in the low Prandtl number case, the temperature elevation under the beam scales with the $\frac{3}{4}$ power, and the Nusselt number behaves like

$$\mathrm{Nu} \sim \mathrm{Ma}^{1/4}\mathrm{Pr}^{1/8}. \tag{23}$$

These relations agree with the results of Chan, Chen, and Mazumder [11], who explicitly solved the equations of motion under the beam. As is clear from our approach, these solutions cannot faithfully represent the dissipation in the entire system, although they do provide the right scaling behavior, and the correct order of magnitude of the numerical prefactor in Eqs. (20,23), which happens to be of order 1. Our numerical solutions in the stationary regime confirmed the scaling laws (19) and (20). This gives us confidence in our approach.

We now consider the problem of determining the Nusselt number when the fluid is turbulent. Precise experimental and numerical results on this problem are not yet available, so one has to make assumptions about the structure of the flow [14]. First, we assume that the large scale flow consists of rolls extending over the entire system, as was the case in the laminar problem. We also assume that the Marangoni effect generates jets of fluid that carry fluid away from the center, and run along the walls, and then concentrate to make a vertical upward moving jet under the heating beam. Contrary to the laminar problem, these jets do not reduce to thin boundary layers under the free surface, since turbulent jets tend to open up with a finite angle as they propagate downstream. As a consequence, the geometry of the jets is independent of the heating (the Marangoni number), and depends only on the aspect ratios of the container. Let U_t be the order of magnitude of the fluid velocity inside these jets. Under the heating beam, the upward moving jet splits to run along the interface, leaving a turbulent stagnation point region. This will confine the heated region right under the beam, in a narrow region of width Δ_t. We expect that this region becomes narrower upon increasing the heating (Marangoni number). We denote by Θ_t the (dimensionless) temperature elevation in the fluid.

The temperature elevation, Θ_t, as well as the velocity in the jets, U_t, and the width of the heated region, Δ_t, are now predicted with the help of these assumptions, and with the method used in the laminar case. We begin by recalling the estimation of the kinetic energy dissipation per unit mass, $\epsilon \sim U^3/L$, where U and L are the (large) velocity and length scales. In our problem, the dissipation of kinetic energy is located in the jets, and it can be estimated as

$$\mathrm{Pr}\sum_{i,j} \int_{\mathrm{vol}} (\partial_i u_j)^2 d^3x \approx U_t^3. \tag{24}$$

Balancing the energy dissipation, Eq. (24), with the production term, Eq. (12), leads to

$$U_t^2 \sim \mathrm{Pr\ Ma}\Theta. \tag{25}$$

The estimation of the thermal dissipation is identical to the laminar case, so Eq. (11) leads to

$$\Theta_t \sim \Delta_t. \tag{26}$$

Finally, a last relation can be obtained by equating the advection term in the thermal boundary layer, $u \cdot \boldsymbol{\nabla}\theta \sim U_t\Theta/\Delta_t$ and the diffusion term, Θ_t/Δ_t^2, leading to

$$\Delta_t U_t \sim 1. \tag{27}$$

Combining Eqs. (25–26) together, one obtains

$$\Theta_t \sim \Delta_t \sim (\mathrm{Pr\ Ma})^{-1/3} \tag{28a}$$

$$U_t \sim (\mathrm{Pr\ Ma})^{1/3}. \tag{28b}$$

In dimensional units, the temperature elevation is therefore proportional to the $\frac{2}{3}$ power of the heating power, or equivalently, the Nusselt number in this case behaves as

$$\mathrm{Nu} \sim \mathrm{Ma}^{1/3}\mathrm{Pr}^{1/3}, \tag{29}$$

Estimating the order of magnitude of the numerical prefactor in Eq. (29) would require a more precise knowledge of the geometry of the flow.

We have estimated the temperature elevation in a weld pool heated by a strong, localized heat flux. In the laminar case, and under the assumption that the flow is in a steady state, we have found that the highest value of the temperature in the fluid scales as the intensity of the heating flux to the power $\frac{3}{4}$, in agreement with the theoretical work of Ref. [11]. With some simple assumptions about the structure of the flow in the turbulent case, we predict that the temperature elevation in the fluid is proportional to the intensity of the heating, to the $\frac{2}{3}$ power. The latter prediction rests on a number of assumptions, which call for more experimental or numerical work. We also emphasize that the assumption of a flat interface quite possibly limits the relevance of this work to practical situations.

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